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Source Localization on Two-Dimensional Grid

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***Abstract*—A computationally efficient algorithm is presented for locating multiple sources on a two-dimensional grid. The total number of the sources as well as their intensities and locations on the two-dimensional grid are assumed unknown and the intensities of the sources are not necessarily identical. The distribution of the source intensity on the grid, which contains all the information about the sources, is obtained by solving a convex optimization problem. With the source distribution interpreted as a two-dimensional gray-level image, the sources are determined as the centroids of the objects in the image. Given the number of sources and the source locations, the source intensities are estimated by solving a linear least-squares problem.**

Keywords: localization; convex optimization; two-dimensional grid; image processing

I. Introduction

The objective of source localization is to determine the locations of the sources as well as relevant source parameters such as the source intensities. Early work on source localization was focused on a single source. Multiple source localization has gained much attention recently [1]–[4]. Most existing source localization methods assume that the number of the sources is known. In this case, source localization is a parameter estimation problem, generally formulated as a least-square estimation problem [5] or a maximum likelihood estimation problem [1]–[3], [6], with the focus on how to find the global minimum of the parameter estimation problem efficiently [1], [2], [4], [7]. When the number of sources is unknown, however, the number of sources must be determined appropriately along with the source locations and other parameters [4], which poses a unique challenge for multiple source localization. In general, the models of possible numbers of sources need to be compared exhaustively and then the best model is selected under a certain model selection criterion, for example, the Bayes Information Criterion [4]. Many model selection criteria exist [8], and the best model depends on the criterion used [8]. All model selection criteria are a compromise between model complexity and model accuracy and involve some design parameters. Here the model complexity is given by the number of independent parameters in the model, proportional to the number of sources. The exhaustive comparison-based multiple source localization method needs to solve a parameter estimation problem under each and every hypothesis and then choose the best hypothesis under a model selection criterion. It is inefficient when the number of sources is large.

This paper presents an estimation method for localizing an unknown number of stationary sources. The key assumption of the method are 1) that the sources are located on a two-dimensional grid and 2) that the sensor observation satisfies the linear superposition principle. The first assumption is valid when the grid has sufficient resolution over the regions of interest. The second assumption means that the sensor observation due to all the sources equals the sum of the contributions of the individual sources to the sensor observation. Most sensors that measure the received energy or power, for example, acoustic sensors that measure the source transmitted energy [1]–[3], [6], [7], satisfy the second assumption. Instead of determining the number of sources as well as the source parameters (locations and intensities) based on exhaustive model comparison, the method of this paper solves for the intensity distribution on the grid directly. Then, the number of sources and the source parameters are inferred from the intensity distribution with a source defined as the neighboring grid points of high intensity concentration. The method is efficient because it avoid computationally expensive model comparison and solves a convex optimization problem for intensity distribution. The convex optimization problem can be solved much efficiently than the original non-convex optimization problem.

The organization of the remainder of the paper proceeds as follows. First, the sensor measurement model is reviewed. Second, the general source localization problem is defined. Then, a convex optimization problem for estimating the intensity distribution on the grid is formulated. Next, the post-processing that infers the source parameters is presented. Finally, numerical results and concluding remarks are shown.

II. Sensor Model

Let M be the number of sensors that measure the power received from the transmitting sources and K the number of sources. In the absence of noise, the received power b_j at the j^{th} sensor due to the K sources is given by [1]

$$b_j = g_j \sum_{k=1}^{K} \frac{\alpha_k}{d_{jk}^{\gamma}(x_j^s, y_j^s; x_k, x_k)} \tag{1}$$

where g_j is the gain of the sensor, α_k is the intensity of the k^{th} source, x_j^s and y_j^s are the coordinates of the j^{th} sensor, x_k and y_k are the unknown coordinates of the k^{th} source; and $d_{jk}(x_j^s, y_j^s; x_k, x_k)$ is the Euclidean distance between the j^{th} sensor and the k^{th} source, defined by

$$d_{jk}(x_j^s, y_j^s; x_k, x_k) = \sqrt{(x_k - x_j^s)^2 + (y_k - y_j^s)^2} \tag{2}$$

and $\gamma \geq 2$ is the path loss exponent. This sensor model satisfies linear superposition. In this paper, we assume $\gamma = 2$ and rewritten (1) as

$$b'_j = g_j \sum_{k=1}^{K} \frac{\alpha_k}{(x_k - x_j^s)^2 + (y_k - y_j^s)^2} \tag{3}$$

The noisy sensor measurement $\tilde{b}'_j$ is given by

$$\tilde{b}'_j = b'_j + \nu'_j \tag{4}$$

where ν'_j is assumed to be additive Gaussian noise with mean μ_j and variance σ'^2_j. The detailed model of ν'_j for acoustic sensors can be found in [1]. It is further assumed that ν'_i and ν'_j, $i \neq j$, are independent of each other. The sensor model may be rewritten as

$$\tilde{b}_j = \frac{\tilde{b}'_j - \mu_j}{g_j} = b_j + \nu_j \tag{5}$$

where ν_j is additive Gaussian noise with zero mean and variance $\sigma_j^2 = \sigma'^2_j / g_j$ and

$$b_j = \sum_{k=1}^{K} \frac{\alpha_k}{(x_k - x_j^s)^2 + (y_k - y_j^s)^2} \tag{6}$$

Equations (5) and (6) define the sensor model used in later sections.

III. Multiple Source Localization

Given the sensor measurements $\tilde{b}_j$ and sensor properties x_j, y_j, and σ_j^2, $j = 1, \ldots, M$, the objective of multiple source localization is to find the source parameters α_k, x_k, y_k, $k = 1, \ldots, K$, and the number of sources K. Under the independence and Gaussian assumption, the likelihood function for the source parameters is

$$L = \prod_{j=1}^{M} L_j, \quad L_j = \frac{1}{\sqrt{2\pi\sigma_j^2}} \exp\left[-\frac{(\tilde{b}_j - b_j)^2}{2\sigma_j^2}\right] \tag{7}$$

The maximum likelihood estimate of the $3K$ source parameters of K sources minimizes the following objective function

$$J' = -\sum_{j=1}^{M} \ln L_j \tag{8}$$

where ln is the natural logarithm and $M \geq 3K$. The source parameter-dependent part of the objective function is given by

$$J = \sum_{j=1}^{M} \frac{1}{2\sigma_j^2} \left[\tilde{b}_j - \sum_{k=1}^{K} \frac{\alpha_k}{(x_k - x_j^s)^2 + (y_k - y_j^s)^2}\right]^2 \tag{9}$$

This optimization problem is difficult to solve because it is non-convex and the number K of sources is unknown.

IV. Source Intensity Distribution on the Grid

Because the sensor model is linear in the intensities, a convex optimization formulation for multiple source localization is possible by discretizing (gridding) the region of feasible source locations. The discretization process allows for direct estimation of the source intensity distribution on the grid, too.

A. Discretization

The space of feasible source location is discretized into a two-dimensional grid of a total of N grid points. The locations (x_i^G, y_i^G), $i = 1, \ldots, N$, of the grid points are known and $N \gg K$. In most cases, $N > M$, too. The two-dimensional intensity distribution on the grid is represented by an $N \times 1$ vector $\mathbf{X}$, given by

$$\mathbf{X} = \begin{bmatrix}\alpha_1^S & \alpha_2^S & \ldots & \alpha_N^S\end{bmatrix}^T \tag{10}$$

The i^{th} element $\alpha_i^S \geq 0$ of $\mathbf{X}$ denotes the intensity of a source located at the i^{th} grid point at (x_i^G, y_i^G). The vector $\mathbf{X}$ representing K point sources is a sparse vector with all but K elements zero. Rewriting the noiseless model given by (6) as a function of α_i^S yields

$$b_j = \sum_{i=1}^{N} \frac{\alpha_i^S}{(x_i^G - x_j^s)^2 + (y_i^G - y_j^s)^2} \tag{11}$$

Note that the only unknowns in the above equation are α_i^S and b_j are linear in α_i^S. In matrix form, the equation is

$$\mathbf{b} = A\mathbf{X} \tag{12}$$

where

$$\mathbf{b} = \begin{bmatrix}b_1 & b_2 & \ldots & b_M\end{bmatrix}^T \tag{13}$$

and

$$A = \begin{bmatrix} \frac{1}{(x_1^G - x_1^s)^2 + (y_1^G - y_1^s)^2} & \cdots & \frac{1}{(x_N^G - x_1^s)^2 + (y_N^G - y_1^s)^2} \\ \vdots & \ddots & \vdots \\ \frac{1}{(x_1^G - x_M^s)^2 + (y_1^G - y_M^s)^2} & \cdots & \frac{1}{(x_N^G - x_M^s)^2 + (y_N^G - y_M^s)^2} \end{bmatrix} \tag{14}$$

The observation (or sensing) matrix A is fully determined by the sensors and the grid. For stationary sensors and fixed grid, the matrix only needs to be computed once off-line. The size of A is $M \times N$. Since $M < N$, the linear system is under-determined. Given A and $\mathbf{b}$, there are an infinite number of $\mathbf{X}$ that satisfy (12) exactly. In the presence of noise, (12) becomes

$$\tilde{\mathbf{b}} = A\mathbf{X} + \boldsymbol{\nu} \tag{15}$$

where

$$\boldsymbol{\nu} = \begin{bmatrix}\nu_1 & \nu_2 & \ldots & \nu_M\end{bmatrix}^T \tag{16}$$

is a zero-mean Gaussian vector with covariance $R = \text{diag}(\sigma_1^2, \ldots, \sigma_M^2)$.

B. Intensity Distribution Estimation

The source intensity distribution estimation problem is now defined as follows: Given A and $\tilde{\mathbf{b}}$, determine the best estimate $\hat{\mathbf{X}}$ of $\mathbf{X}$. Because $\mathbf{X}$ is sparse, we require that $\hat{\mathbf{X}}$ be close to being sparse, too. The estimate $\hat{\mathbf{X}}$ will be obtained by solving a convex optimization problem (ℓ_1 minimization problem [9]).

If there were no measurement noise or the measurement noise is negligibly small, the optimal estimate $\hat{\mathbf{X}}$ could be defined as the solution to an ℓ_1 minimization problem [9]:

$$\hat{\mathbf{X}} = \arg\min_{\mathbf{X}} \|D\mathbf{X}\|_1 \tag{17}$$

subject to

$$A\mathbf{X} = \tilde{\mathbf{b}},\ \mathbf{0} \leq \mathbf{X} \leq \mathbf{X}^u \tag{18}$$

where D is a diagonal matrix with the diagonal elements given by the 2-norms of the columns of A. The 2-norm of a vector is defined by

$$\|\mathbf{X}\|_2 = \sum_{i=1}^{N} |X_i|^2 \tag{19}$$

The constraint $\mathbf{0} \leq \mathbf{X} \leq \mathbf{X}^u$ requires that all the elements of $\hat{\mathbf{X}}$ be bounded by the lower bound $\mathbf{0}$ and upper bound $\mathbf{X}^u$. The 1-norm of a vector is defined by

$$\|\mathbf{X}\|_1 = \sum_{i=1}^{N} |X_i| \tag{20}$$

The 1-norm is an approximation to the 0-norm, which is the number of nonzero elements of a vector [9]. The sparsity of a vector is measured by the 0-norm of the vector. The ℓ_1 minimization problem is equivalent to

$$\hat{\mathbf{X}}' = \arg\min_{\mathbf{X}'} \|\mathbf{X}'\|_1 \tag{21}$$

subject to

$$\hat{A}\mathbf{X}' = \tilde{\mathbf{b}},\ \mathbf{0} \leq \mathbf{X}' \leq D\mathbf{X}^u \tag{22}$$

with $\hat{A} = AD^{-1}$ and $\hat{\mathbf{X}}' = D\hat{\mathbf{X}}$. The columns of $\hat{A}$ are unit vectors but are not necessarily perpendicular to each other. The idea of ℓ_1 minimization is to represent $\tilde{\mathbf{b}}$ by the minimum number of columns of $\hat{A}$.

The above optimization problem is now modified to account for measurement noise. Although the system is underdetermined, i.e., $N > M$, there may be no nonnegative $\mathbf{X}$ satisfying the equality constraint $\tilde{\mathbf{b}} = A\mathbf{X}$. Even if the equality constraint can be satisfied, the solution may not be statistically optimal. So in the presence of noise, the optimal solution should

1) be nonnegative;
2) minimize the residual error $\tilde{\mathbf{b}} - A\mathbf{X}$;
3) be as sparse as possible.

The second and third objectives are conflicting with each other. Delicate balance between the two objectives is therefore required for the optimal solution. Our formulation of the optimization problem in this case is as follows:

$$\hat{\mathbf{X}} = \arg\min_{\mathbf{X}} \|D\mathbf{X}\|_1 \tag{23}$$

subject to

$$(A\mathbf{X} - \tilde{\mathbf{b}})^T R^{-1} (A\mathbf{X} - \tilde{\mathbf{b}}) \leq \lambda, \quad \mathbf{0} \leq \mathbf{X} \leq \mathbf{X}^u \tag{24}$$

where λ is a control parameter. The lower bound $\lambda_{\min}$ of λ can be found by solving

$$\bar{\mathbf{X}} = \arg\min_{\mathbf{X}} (A\mathbf{X} - \tilde{\mathbf{b}})^T R^{-1} (A\mathbf{X} - \tilde{\mathbf{b}}) \tag{25}$$

subject to

$$\mathbf{0} \leq \mathbf{X} \leq \mathbf{X}^u \tag{26}$$

Then,

$$\lambda_{\min} = (A\bar{\mathbf{X}} - \tilde{\mathbf{b}})^T R^{-1} (A\bar{\mathbf{X}} - \tilde{\mathbf{b}}) \tag{27}$$

We may choose $\lambda = \kappa\lambda_{\min}$, where $\kappa > 1$. When $\bar{\mathbf{X}}$ is the true source intensity distribution, $(A\bar{\mathbf{X}} - \tilde{\mathbf{b}})^T R^{-1} (A\bar{\mathbf{X}} - \tilde{\mathbf{b}})$ obeys the χ^2 distribution of M degrees of freedom and of mean M and standard deviation $\sqrt{2M}$. So, another choice of λ is $\lambda = M + K\sqrt{2M}$, with K a free parameter.

`CVX` for MATLAB, a package for specifying and solving convex programs [10], [11], is used to solve the convex optimization problems. The exit conditions of the `CVX` solver include "Solved," "Inaccurate/Solved," "Unbounded," "Inaccurate/Unbounded," "Infeasible," "Inaccurate/Infeasible," "Failed," and "Overdetermined." We accept the solution when the condition is either "Solved" or "Inaccurate/Solved."

C. Post Processing

The estimate $\hat{\mathbf{X}}$ gives the source intensity distribution on the grid. All information about the sources is contained in $\hat{\mathbf{X}}$. When there is no noise in the sensor data, and the estimate $\hat{\mathbf{X}}$ is a sparse vector that reconstructs the true source intensity distribution, only a small number of the elements of $\hat{\mathbf{X}}$ are nonzero (positive). The number of sources is given by the number of positive elements of $\hat{\mathbf{X}}$. The source intensities are the positive intensity elements of $\hat{\mathbf{X}}$. The locations of the sources are determined by examining the indices of the positive elements. Suppose the i^{th} element of $\hat{\mathbf{X}}$ or equivalently the i^{th} grid point corresponds to a source, the location of the source is (x_i^G, y_i^G).

In the presence of noise, the estimate $\hat{\mathbf{X}}$ is less sparse, having more elements that are much greater than zero. In most cases, a single source is associated with a number of spatially adjacent grid points in a small area (a cluster). That is, a single point source is given by an area of high concentration in the estimated intensity distribution.

To determine the sources using image processing methods, the first step is to convert the one-dimensional vector $\hat{\mathbf{X}}$ to a two-dimensional gray-level image of which the gray levels at (x_i^G, y_i^G) are $\hat{X}_i$. Then, the number of sources and their locations are determined using the procedure below:

1) Find the Otsu threshold [12] to filter the background noise of the image. The Otsu method is used to determine the appropriate threshold to discriminate the noise from the true sources over the grid. The basic idea of the method is that the image to be thresholded contains two classes of pixels which should be distinct with respect to the intensity values of their pixels. This method shows optimal performance in the sense that it maximizes the between-class variance, a well-known measure used in statistical discriminant analysis.
2) Find out the edges of the filtered image using the MATLAB function `edge`. An edge detector is used to capture the intensity changes in the images. All edge detectors are based on convolving the image with a small, separable, and integer valued filter in horizontal and vertical directions. They differ in how to approximate the gradients. Generally, the edge detectors use two kernels which are convolved with the original image to calculate approximations of the gradients in two directions, one

for horizontal changes and one for vertical. Let $\mathcal{X}$ be the source image and $\mathcal{G}_x$ and $\mathcal{G}_y$ be two images that at each point contain the horizontal and vertical derivative approximations. The gradient images convolved with the Prewitt operator are computed as:

$$\mathcal{G}_x = \begin{bmatrix} -1 & 0 & 1 \\ -1 & 0 & 1 \\ -1 & 0 & 1 \end{bmatrix} \otimes \mathcal{X}, \quad \mathcal{G}_y = \begin{bmatrix} -1 & -1 & -1 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \otimes \mathcal{X} \tag{28}$$

where $\otimes$ denotes the convolution operator. The gradient images convolved with the Sobel operator are computed as:

$$\mathcal{G}_x = \begin{bmatrix} -1 & 0 & 1 \\ -2 & 0 & 2 \\ -1 & 0 & 1 \end{bmatrix} \otimes \mathcal{X}, \quad \mathcal{G}_y = \begin{bmatrix} -1 & -2 & -1 \\ 0 & 0 & 0 \\ 1 & 2 & 1 \end{bmatrix} \otimes \mathcal{X} \tag{29}$$

The Sobel method is used in this paper. The outcome of this step is a binary images of the same size as $\mathcal{X}$.

3) Find the centroids of the connected components in the binary image using the MATLAB function `regionprops`. Each connected component in the image represents a source. Hence, the number of the sources is the number of connected components; the locations of the sources are the centroids of the connected components. Weighted centroids may be used if the original gray-level image is processed in this step.
4) Compute the intensity of the sources by solving a standard linear least-square problem. Given the number K of sources and their locations $(x_k,\ y_k)$, the source intensities α_k are the solution to the least-square problem with the following cost function

$$J = \sum_{j=1}^{M} \frac{1}{2\sigma_j^2} \left[\tilde{b}_j - \sum_{k=1}^{K} \frac{\alpha_k}{(x_k - x_j^s)^2 + (y_k - y_j^s)^2} \right]^2 \tag{30}$$

This method is more accurate than computing the intensity of a source as the sum of the intensities of the connected component of the image corresponding to the source.

V. Numerical Result

The region of interest is a 30 m by 30 m rectangle. A uniform 50×50 grid is generated over the region. So, the number of grid points $N = 2500$. The numbers of the stationary sensors that cover the region are $M = 100$. The same coverage may be achieved by much fewer mobile sensors with known position and movement. The sensors are uniformly randomly deployed to provide good coverage. The matrix A is determined by the sensor locations and the grid. Its size is 100×2500. The maximum number of sources in the region is 5. The true source locations are randomly selected from the 2500 grid points. The intensity of a source is randomly chosen between 500 and 1000. The noise variance is 1. Because of the length limitation of the paper, only one representative numerical result of 5 sources is shown in detail. The true source distribution is shown in Fig. 1, where the default colormap is used. The source intensities are 600, 800, 1000, 800, 1000, respectively. The estimated source distribution

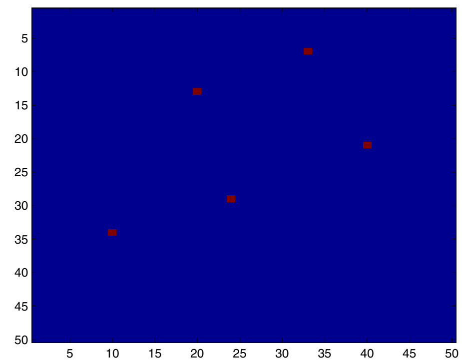

Figure 1. True Source Distribution

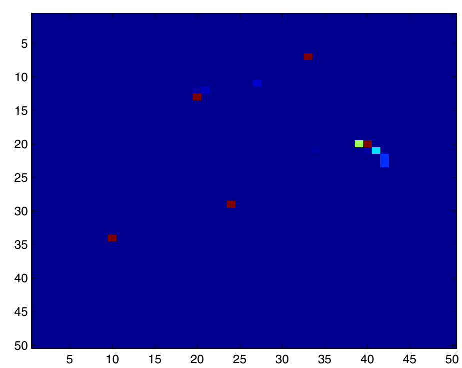

Figure 2. Estimated Source Distribution

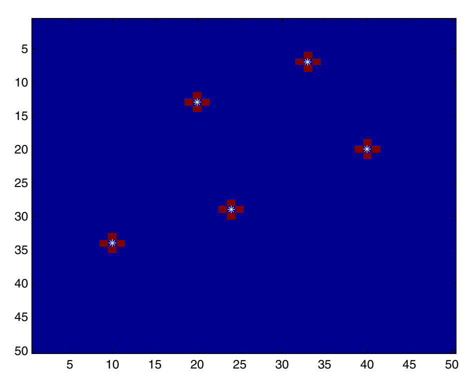

Figure 3. Estimated Source Distribution after Post-Processing

is shown in Fig. 2. All the information about the sources are contained there. The sources are the five distinctive objects

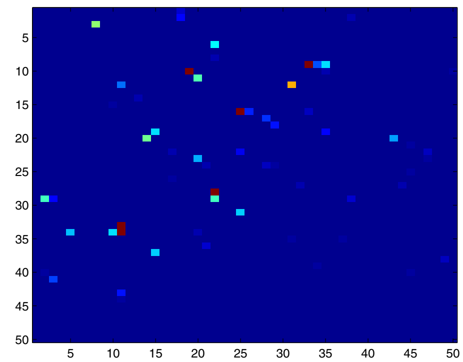

Figure 4. Estimated Source Distribution (Large Sensor Noise)

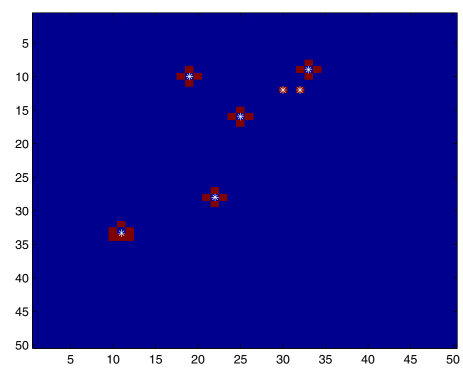

Figure 5. Estimated Source Distribution after Post-Processing (Large Sensor Noise)

in the image. The source distribution after post processing is shown in Fig. 3. Comparing Fig. 1 and Fig. 3, the locations of four of the five sources are exactly reconstructed. The localization error is one pixel or 0.6 meters in the y coordinate of the rightmost source. The estimated intensities are 1001.1, 800.5, 801.4, 598.3, 904.5, respectively. The largest error, 5.5, occurs at the rightmost source, too. Monte Carlo tests indicate that the method is accurate and robust in the presence of Gaussian noise of variance one.

When the noise variance increases to 100 but the number of sensors remains 100, the estimated intensity distribution becomes highly inaccurate. This can be seen from Fig. 4, where there are too many weak sources. The post processing can still retrieve the source information from the estimated intensity distribution, but the result is not in agreement with the true source distribution in terms of the number of sources and the source parameters (see Fig. 5). The rightmost true source is not identified by post processing because in Fig. 4 there exists no significant source in the neighborhood of the rightmost true source. In Fig. 5, there are two weak sources denoted by $*$ and one true source is missing. The estimates of the source locations and intensities are poor. However, that is due to the low SNR in this case.

Finally, we note that the multiple source localization method is computationally efficient. The total execution time of the method on a MacBook Pro computer with an Intel Core 2 Duo processor, including the execution time for the computation of the matrix A, which only needs to be computed once, is less than one minute.

VI. Concluding Remarks

A computationally efficient method for two-dimensional source localization was presented. A key idea of the method is to estimate the two-dimensional source intensity distribution directly, which provides the complete picture about the distribution of the sources. Image processing methods are then used to retrieve information about the sources from the image, with the sources being areas of high intensity concentration. The use of a high resolution grid by the method leads to a large scale optimization problem, but because the optimization problem is convex, it can be solved efficiently and the solution is guaranteed to be globally optimal. The representative example shows that the accuracy of the source localization solution depends on the resolution of the grid, the number of sensors, and the SNR and that the source localization method yields reliable results.

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