Asymptotic Connectivity in Wireless Ad Hoc Networks Using Directional Antennas

Pan Li, Chi Zhang, *Student Member, IEEE,* and Yuguang Fang, *Fellow, IEEE*

Abstract—**Connectivity is a crucial issue in wireless ad hoc networks (WANETs). Gupta and Kumar have shown that in WANETs using omnidirectional antennas, the critical transmission range to achieve asymptotic connectivity is $O(\sqrt{\log n/n})$ if n nodes are uniformly and independently distributed in a disk of unit area. In this paper, we investigate the connectivity problem when directional antennas are used. We first assume that each node in the network randomly beamforms in one beam direction. We find that there also exists a critical transmission range for a WANET to achieve asymptotic connectivity, which corresponds to a critical transmission power (CTP). Since CTP is dependent on the directional antenna pattern, the number of beams, and the propagation environment, we then formulate a non-linear programming problem to minimize the CTP. We show that when directional antennas use the optimal antenna pattern, the CTP in a WANET using directional antennas at both transmitter and receiver is smaller than that when either transmitter or receiver uses directional antenna and is further smaller than that when only omnidirectional antennas are used. Moreover, we revisit the connectivity problem assuming that two neighboring nodes using directional antennas can be guaranteed to beamform to each other to carry out the transmission. A smaller critical transmission range than that in the previous case is found, which implies smaller CTP.**

Index Terms—**Wireless ad hoc networks, directional antenna, asymptotic connectivity, critical transmission range, critical transmission power**

I. Introduction

FOR a long time, people have dreamed of breaking through the limitation of physical distance and communicating with each other tetherlessly anytime at any place. Now, this dream is being realized step by step with the rapid development and deployment of wireless networks, such as wireless local area networks (WLANs), wireless ad hoc networks (WANETs) including mobile ad hoc networks (MANETs) and wireless sensor networks (WSNs) [1], and wireless mesh networks (WMNs) [2]. Although omnidirectional antennas are commonly used in these networks, directional antennas have gained tremendous attention due to the increased transmission range, the improved spatial reuse, and the decreased interference.

Up to now, many research works on wireless networks using directional antennas focus on the design of MAC protocols ([3]–[13]). There are relatively few works on network connectivity, which is indeed one important problem. Gupta and Kumar [14] study the connectivity problem for wireless networks using omnidirectional antennas and show that there is a critical transmission range $O(\sqrt{\log n/n})$ for a network to achieve asymptotic connectivity when there are n nodes uniformly and independently distributed in a disk of unit area. This problem has not been studied when directional antennas are used except in our preliminary work [15].

In this paper, we address the connectivity problem in WANETs using directional antennas. Other than using the simple sector model, we introduce a more realistic directional antenna model, which consists of one main lobe with main lobe antenna gain G_m, as well as $N - 1$ ($N > 1$) side lobes with the same side lobe antenna gain G_s. We show that the side lobe antenna gain does have a significant impact on the network connectivity, which cannot be simply neglected. In our model, directional antennas can work either in the directional mode, or in the omnidirectional mode. So, according to the usage of directional antennas, we can classify WANETs using directional antennas into four categories: *DTDR (Directional Transmission and Directional Reception) networks, DTOR (Directional Transmission and Omnidirectional Reception) networks, OTDR (Omnidirectional Transmission and Directional Reception) networks and OTOR (Omnidirectional Transmission and Omnidirectional Reception) networks.* Note that OTOR networks are exactly the networks Gupta and Kumar study in [14]. We also note that for directional antennas, with fixed transmission power, the directional transmission range is directly dependent on the omnidirectional transmission range given the directional antenna pattern (G_m, G_s), the number of beams N, and the path loss exponent α.

Assuming every node in a WANET randomly beamforms in one beam direction, we derive the necessary and sufficient conditions for the WANET to achieve asymptotic connectivity. We find that there also exists a critical omnidirectional transmission range. Thus, we can compare the power consumption when directional antennas are used with that when omnidirectional antennas are used, i.e., in OTOR (Omnidirectional Transmission and Omnidirectional Reception) networks, by simply comparing their critical omnidirectional transmission ranges. For simplicity, we call *critical transmission range* (CTR) instead.

Specifically, we show that compared to the critical transmission range $r_0(n)$ in OTOR networks, the critical transmission

range is $a_1^{-\frac{1}{2}} r_0(n)$ in DTDR networks, $a_2^{-\frac{1}{2}} r_0(n)$ in DTOR networks, and $a_3^{-\frac{1}{2}} r_0(n)$ in OTDR networks, respectively, where $a_1 = [\frac{1}{N}(G_m)^{\frac{2}{\alpha}} + \frac{N-1}{N}(G_s)^{\frac{2}{\alpha}}]^2$, and $a_2 = a_3 = [\frac{1}{N}(G_m)^{\frac{2}{\alpha}} + \frac{N-1}{N}(G_s)^{\frac{2}{\alpha}}]$. When $a_1 > 1$, the critical transmission range in DTDR networks is smaller than that in OTOR networks. So, the corresponding transmission power for achieving asymptotic connectivity, which we call *critical transmission power* (CTP), in DTDR networks is also smaller than that in OTOR networks. DTOR (or OTDR) networks have the same situation if a_2(or a_3) > 1. Then, a question arises: how low can the critical transmission power level go?

In order to minimize the CTP in DTDR, DTOR, and OTDR networks, respectively, we formulate a non-linear programming problem to find the optimal directional antenna pattern (G_m, G_s), given the number of beams N and the path loss exponent α. We also investigate the impacts of N and α on the network connectivity. We show that in the outdoor environments $(\alpha \in [2, 5])$, with the same antenna pattern (G_m, G_s), and the same number of beams $N(N > 1)$, when the CTP in DTOR or OTDR networks is smaller than that in OTOR networks, the CTP in DTDR networks is further smaller than that in DTOR or OTDR networks. However, when the CTP in DTOR or OTDR networks is larger than that in OTOR networks, the CTP in DTDR networks is further larger than that in DTOR or OTDR networks. Moreover, with the same N and α, the minimum CTP, which is achieved when directional antennas use the optimal antenna pattern, in DTDR networks is always smaller than that in DTOR and OTDR networks, which is further smaller than that in OTOR networks when $N > 2$. When $N = 2$, the minimum CTP in DTDR networks is the same as that in DTOR, OTDR, and OTOR networks. Besides, as N increases, or α decreases, the minimum CTP decreases, which means we can save more power.

Furthermore, we revisit the connectivity problem in the more relaxed sense that there exists a schedule according to which two neighboring nodes can be guaranteed to beamform to each other to carry out the transmission. A smaller critical transmission range than that in the previous case is found, which implies smaller CTP.

The rest of this paper is organized as follows. We review the related work in the next section. In Section III, we introduce our directional antenna model and the power propagation model. Section IV presents the necessary and sufficient conditions for WANETs using directional antennas to achieve asymptotic connectivity. In Section V, we illustrate how to find the optimal directional antenna pattern in order to attain the asymptotic connectivity with the smallest transmission power. We also discussed the impacts of the number of beams and the path loss exponent on the network connectivity. Section VI gives some insights on our results. An extended case is studied in Section VII. We finally conclude this paper in Section VIII.

II. RELATED WORK

How to achieve connectivity is a very important problem in WANETs. Gupta and Kumar [14] show that for a network of n uniformly and independently distributed nodes to achieve asymptotic connectivity, there is a CTR required for

connectivity, which is $O(\sqrt{\log n/n})$. While the work in [14] assumes the nodes in the network are static, Madsen et al. [16] study the connectivity probability in mobile ad hoc networks, where the position of the nodes and the link quality change over time. Some stationary mobility models are considered there, including random direction model, random waypoint model, attractor model, virtual world model, and mobility models with obstacles. Dousse et al. [17] study the impact of interferences on connectivity in ad hoc networks and show that there is a critical value of γ above which the network is made of disconnected clusters of nodes, where γ is a coefficient that weighs the effect of interferences. Recently, Yi et al. investigate the asymptotic connectivity problem for ad hoc networks with Bernoulli nodes in [18] and [19].

Other than asymptotic connectivity discussed in the works above, Balister et al. [20] derive for the first time reliable density estimates for thin strip networks with limited number of nodes to achieve connectivity. They also demonstrate the accuracy of the estimates through simulations. [20] represents a very useful work for practical deployment and bridges the gap between theory and practice. Since we are envisioning a wireless network with a very large number of nodes in the future, we will focus on asymptotic connectivity in this paper as that in [14].

Notice that all the works above assume the use of omnidirectional antennas in the networks. There are also some works related to connectivity for networks using directional antenna. Kranakis et al. [21] study the k-connectivity problem in wireless sensor networks and derive sufficient conditions on the beamwidth of directional antennas so that the energy consumption required to maintain k-connectivity is lower when using directional antennas than when using omnidirectional antennas. Diaz et al. [22] investigate the value of the chromatic number $\chi(G_n)$, the directed clique number $\omega(G_n)$, and the undirected clique number $\widehat{\omega_2}(G_n)$ for random scaled sector graphs[1]. They show that when $\alpha < \pi$, where α is the beam width of directional antennas, $\chi(G_n)$ and $\widehat{\omega_2}(G_n)$ are $\Theta(\frac{\ln n}{\ln \ln n})$ with high probability (w.h.p.), while $\omega(G_n)$ is $O(1)$ w.h.p., as $n \to +\infty$, and when $\alpha > \pi$, w.h.p. $\chi(G_n)$ and $\widehat{\omega_2}(G_n)$ are $\Theta(\ln n)$, while $\omega(G_n)$ is $\Theta(\frac{\ln n}{\ln \ln n})$. Besides, Bettstetter et al. [23] examine the impact of randomized beamforming on multihop wireless networks via simulations.

However, these researches have not taken the transmission and the reception schemes into consideration, which are actually very important when we discuss the connectivity problem. Similar to that in [24] where capacity of wireless networks using directional antennas is discussed, we classify in this paper WANETs using directional antennas into four types according to their transmission and reception schemes, i.e., DTDR networks, DTOR networks, OTDR networks, and OTOR networks. We study the CTR in these networks based on a more realistic directional antenna model with both main lobe antenna gain and non-negligible side lobe antenna gain. Then, for each kind of network, we formulate a non-linear

---

[1]The chromatic number, the directed clique number, and the undirected clique number of a graph G are the least number of colors needed to color the graph G, the size of the maximum directed clique in graph G, and the size of the maximum undirected clique in graph G, respectively.
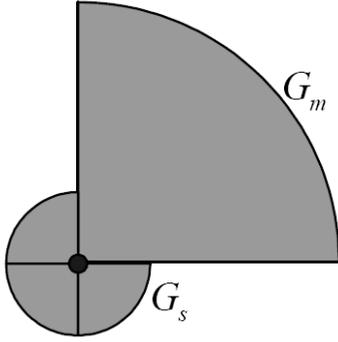
Fig. 1.   Our switched beam directional antenna model.

Fig. 2.   Illustration for calculating the main lobe and the side lobe antenna gain.

programming problem to find the optimal antenna pattern (G_m, G_s), given the number of beams N and the path loss exponent α, in order to provide asymptotic network connectivity with the minimum CTP. We also examine the impact of N and α on network connectivity.

III. PRELIMINARIES

A. Directional Antenna Model

Denote a vector in three-dimensional space by $\vec{d}$ $(\vec{d} \in R^3)$, then the gain of an antenna in the direction $\vec{d}$ is given by [25]:

$$G\left(\vec{d}\right) = \eta \cdot \frac{U(\vec{d})}{U_{ave}},$$

where $U(\vec{d})$ is the power density in the direction $\vec{d}$, U_{ave} is the average power density over all directions, and η $(0 < \eta \le 1)$ is the efficiency of the antenna which accounts for losses. Clearly, we can see that an omnidirecitonal antenna has a gain of 0 dB and a directional antenna can have a higher gain than that. Due to the higher gain and less interference when it is beamforming in a specific constrained direction, a directional antenna can offer a longer transmission distance than an omnidirectional antenna.

In the current literature, there are three primary types of directional antenna systems: the switched beam antenna system, the steered beam antenna system, and the adaptive antenna system [26]. In this study, we use the switched beam antenna system, which consists of several highly directive, fixed, pre-defined beams and each transmission uses only one of the beams. Our study assumes that every antenna has N $(N > 1)$ beams exclusively and collectively covering all directions. The main lobe antenna gain is denoted by G_m and the side lobe antenna gain is denoted by G_s. We assume G_m and G_s are constants in the main lobe direction and side lobe directions, respectively. One such example with four beam directions is shown in Fig. 1. Moreover, directional antennas work either in the directional mode ($0 \le G_s < 1 < G_m$) or in the omnidirectional mode ($G_s = G_m = 1$).

Let P be the transmission power, and S the surface area of the sphere with center at the transmitter and radius R. As shown in Fig. 2, the surface area A on the sphere for a beamwidth of θ is $2\pi Rh$, where h is $R(1 - \cos\frac{\theta}{2})$. By the definition of antenna gain, when we neglect the side lobe gain
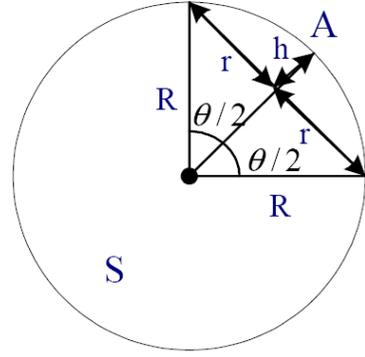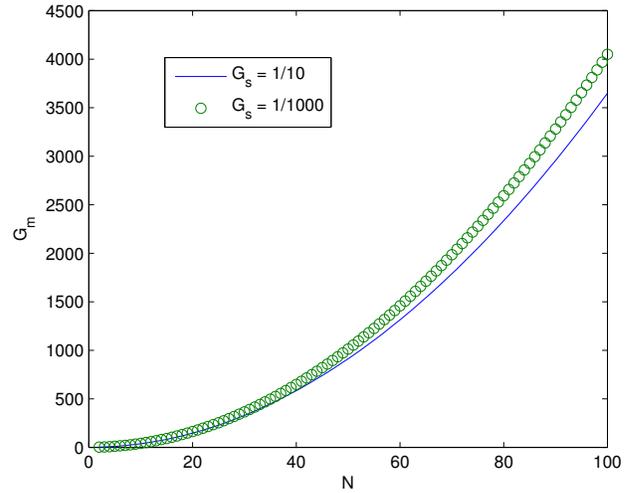
Fig. 3.   The main lobe antenna gain with $N \in [2, 100]$, and $G_s = 1/10$ or $G_s = 1/1000$.

G_s, we have [27]

$$G_m = \frac{P/A}{P/S} = \frac{4\pi R^2}{2\pi R^2(1 - \cos\frac{\theta}{2})} = \frac{2}{1 - \cos\frac{\theta}{2}},$$

and when we consider the side lobe gain G_s, we have

$$G_m \cdot U_{ave} \cdot A + G_s \cdot U_{ave} \cdot (S - A) = \eta \cdot P, \qquad (1)$$

where G_m and G_s are the main lobe directional antenna gain and the side lobe directional antenna gain, respectively. Since $P = S \cdot U_{ave}$, (1) can be simplified as

$$G_m \cdot A + G_s \cdot (S - A) = \eta \cdot S, \qquad (2)$$

from which we can see that there is a relationship among the main lobe antenna gain G_m, the side lobe antenna gain G_s, and the number of beam directions N, which is related to the beam width θ by $N = 2\pi/\theta$. Some of the values derived according to these equations by setting $\eta = 1$ are shown in Fig. 3.

B. Power Propagation Models

Power propagation models are used to predict the received signal strength. A general model is given in (3):

$$P_r(d) = P_t h(h_t, h_r, L, \lambda) \frac{G_t G_r}{d^\alpha}, \tag{3}$$

where P_t and P_r are the transmission power and the reception power, respectively, G_t and G_r are the gain factors for the transmitter antenna and the receiver antenna, respectively, h_t and h_r are the antenna heights of a transmitter and a receiver, respectively, d is the distance between the transmitter and the receiver, L is the system loss factor not related to propagation ($L \geq 1$), λ is the wavelength, $h(\cdot)$ is a function, and α is the path loss exponent.

Two specific models are shown in the following [28], with the path loss exponent equal to 2, and 4, respectively.

The free space propagation model, which is used when the transmitter and receiver have a clear, unobstructed line-of-sight path between them, is given below:

$$P_r(d) = \frac{P_t G_t G_r \lambda^2}{(4\pi)^2 d^2 L}.$$

The two-ray ground reflection model, which considers both the direct path and a ground reflected path between the transmitter and the receiver, is as follows:

$$P_r(d) = P_t G_t G_r \frac{h_t^2 h_r^2}{d^4}.$$

From the above, we can see that the path loss exponent α can be used to characterize the propagation environment. Besides, the value of α usually ranges from 2 to 4 in outdoor environments [28].

IV. NECESSARY AND SUFFICIENT CONDITIONS FOR ASYMPTOTIC CONNECTIVITY

In this section we present the necessary and sufficient conditions for achieving asymptotic connectivity in WANETs using directional antennas. As we mentioned earlier, there are four categories of WANETs in terms of the transmission and the reception schemes: DTDR networks, DTOR networks, OTDR networks, and OTOR networks. We will discuss the connectivity problem in the first three kinds of networks, respectively, in the following three subsections. The case in OTOR networks is the same as that discussed in [14].

We first give the assumptions we use in this paper:

(A1) There are n static nodes uniformly and independently distributed in a disk of unit area on the plane.

(A2) All nodes are equipped with the same switched beam directional antennas. The number of beams is N, the main lobe antenna gain is G_m, and the side lobe antenna gain is G_s.

(A3) All nodes have the same transmission power and transmission range.

(A4) Each node randomly beamforms in one of the N directions, with a probability of $\frac{1}{N}$.

(A5) Edge effects are neglected.

We denote the resulting network graph by $G(V, E(g))$, where V is the vertex set, and $E(g)$ is the edge set defined
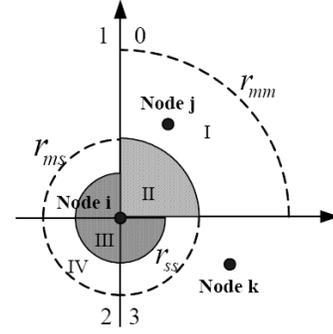
Fig. 4. The communication area of a node i in DTDR networks when directional antennas have four beam directions.

by a function g: $R^2 \to [0, 1]$. Notice that g depends on the transmission and reception schemes. So we use g_1 in DTDR networks, g_2 in DTOR networks and g_3 in OTDR networks.

A. DTDR Networks

We first derive the necessary condition for achieving asymptotic connectivity in DTDR networks. Let r_0 denote the omnidirectional transmission range with the transmission power P_t. With the same transmission power, let r_{mm} denote the transmission range when two nodes beamform to each other, r_{ms} the transmission range when only one of the two nodes beamforms to the other, and r_{ss} the transmission range when neither of the two nodes beamforms to the other. According to the power propagation model in (3), we have

$$r_{mm} = (G_m G_m)^{\frac{1}{\alpha}} r_0, \tag{4}$$

$$r_{ms} = (G_m G_s)^{\frac{1}{\alpha}} r_0, \tag{5}$$

$$r_{ss} = (G_s G_s)^{\frac{1}{\alpha}} r_0. \tag{6}$$

Fig. 4 shows the communication area of an arbitrary node i in DTDR networks. Let r denote the distance to node i, Area I and Area II the areas where $r_{ms} < r \leq r_{mm}$ and $0 < r \leq r_{ms}$ in the main lobe direction, respectively, and Area III and Area IV the areas where $0 < r \leq r_{ss}$ and $r_{ss} < r \leq r_{ms}$ in the side lobe directions, respectively. We observe that:

(1) Each node can communicate with the neighbors in Area I (S_1^{DD}) with probability p_1^{DD}, where

$$p_1^{DD} = \frac{1}{N},$$

$$S_1^{DD} = \frac{1}{N}(\pi r_{mm}^2 - \pi r_{ms}^2).$$

(2) Each node can communicate with the neighbors in Area II (S_2^{DD}) with probability p_2^{DD}, where

$$p_2^{DD} = 1,$$

$$S_2^{DD} = \frac{1}{N}\pi r_{ms}^2.$$

(3) Each node can communicate with the neighbors in Area III (S_3^{DD}) with probability p_3^{DD}, where

$$p_3^{DD} = 1,$$

$$S_3^{DD} = \frac{N-1}{N}\pi r_{ss}^2.$$

(4) Each node can communicate with the neighbors in Area IV (S_4^{DD}) with probability p_4^{DD}, where

$$
\begin{aligned}
p_4^{DD} &= \frac{1}{N}, \\
S_4^{DD} &= \frac{N-1}{N}(\pi r_{ms}^2 - \pi r_{ss}^2).
\end{aligned}
$$

Thus, the *effective area* of nodes in DTDR networks, denoted by S^{DD}, is

$$
\begin{aligned}
S^{DD} &= \sum_{i=1}^{4} p_i^{DD} S_i^{DD} \\
&= \frac{1}{N^2}\pi r_{mm}^2 + \frac{2(N-1)}{N^2}\pi r_{ms}^2 + \frac{(N-1)^2}{N^2}\pi r_{ss}^2 \\
&= \left[\frac{1}{N}(G_m)^{\frac{2}{\alpha}} + \frac{N-1}{N}(G_s)^{\frac{2}{\alpha}}\right]^2 \pi r_0^2.
\end{aligned}
$$

Let $a_1 = [\frac{1}{N}(G_m)^{\frac{2}{\alpha}} + \frac{N-1}{N}(G_s)^{\frac{2}{\alpha}}]^2$. Then, $S^{DD} = a_1 \pi r_0^2$.

Denote the probability that two nodes, i and j, which also denote the positions of these two nodes in R^2, in DTDR networks are connected by $g_1(i,j)$. As shown in Fig. 4, we notice that $g_1(i,j)$ and $g_1(i,k)$ are correlated, where k is another neighboring node of i. So, it is very difficult and complex to directly derive $g_1(i,j)$ analytically. However, as pointed out in [29], we can approximate the connectedness function of an anisotropic system simply by that of an spherical reference system denoted by $g_1(x)$, i.e.,

$$
g_1(x) = \langle g_1(i,j) \rangle = \begin{cases} 1 & \text{if } ||x||_2 \leq \sqrt{a_1} r_0 \\ 0 & \text{if } ||x||_2 > \sqrt{a_1} r_0 \end{cases}, \quad (7)
$$

where $x = x_i - x_j$ denotes the distance between two nodes i and j, the angular brackets indicate an orientation average. Note that [29] shows this approximation is exact when the system is either slightly anisotropic or very anisotropic. Besides, when the system is neither slightly anisotropic nor very anisotropic, this formalism is still capable of describing the connectivity of randomly distributed particle systems over a wide range of particle anisotropy. Thus, in our case, we can use $g_1(x)$ to determine the edge set in the graph $G(V, E(g_1))$, especially when the beam number N goes large[2].

In the following, we use $r_0(n)$ to replace r_0 indicating that the transmission range is dependent on the number of nodes n in order to achieve connectivity. We also denote the probability that $G(V, E(g_1))$ is disconnected by $P_d(n, r_0(n))$. Gupta and Kumar has shown [14] that in networks with omnidirectional antennas, if $\pi r_0(n)^2 = \frac{\log n + c(n)}{n}$, then $P_d(n, r_0(n)) > 0$ when $\limsup_{n \to +\infty} c(n) < +\infty$. Similarly, we can have the following result.

Theorem 1: (i) In DTDR networks, if $a_1 \pi r_0(n)^2 = \frac{\log n + c(n)}{n}$, then

$$
\liminf_{n \to +\infty} P_d(n, r_0(n)) \geq e^{-c}(1 - e^{-c}),
$$

where $c = \lim_{n \to +\infty} c(n)$.
(ii) In DTDR networks, if $a_1 \pi r_0(n)^2 = \frac{\log n + c(n)}{n}$ and

---

[2]As we will see later, directional antennas do need to have very small side lobes in order to achieve connectivity with low transmission power. In particular, the side lobe gain G_s needs to approach 0 to have optimum pattern as the beam number N increases.

$\limsup_{n \to +\infty} c(n) < +\infty$, then $G(V, E(g_1))$ is asymptotically disconnected with positive probability.

Next, we derive the sufficient condition for achieving asymptotic connectivity in DTDR networks using a different approach from that in [14]. We will need some results from continuum percolation by Penrose [30] [31]. Consider a graph where nodes are distributed according to a homogeneous Poisson process in R^2 with intensity λ. We denote this graph by $G^{Poisson}(V, E(g_1))$, where V is the vertex set and $E(g_1)$ is the edge set defined by function g_1 shown in (7). In addition to that, we place a node at the origin $\mathbf{0}$. Then, the resulting point process is a Poisson process "conditioned to have a point at $\mathbf{0}$ in the sense of Palm measures" [30]. We denote this new graph by $G^{Poisson}(V', E(g_1))$, where $V' = V \cup \{\mathbf{0}\}$.

We define the connected components of $G^{Poisson}(V', E(g_1))$ as *clusters*. Let $p_k(\lambda, r(\lambda))$ denote the probability that the cluster containing the origin has k nodes. The percolation probability, denoted by $p_{+\infty}(\lambda, r(\lambda))$, is the probability that $\mathbf{0}$ lies in an infinite cluster when $\lambda \to +\infty$, i.e.,

$$
p_{+\infty}(\lambda, r(\lambda)) = 1 - \sum_{k=1}^{+\infty} p_k(\lambda, r(\lambda)).
$$

Since the function $g_1(x)$ satisfies $g_1(x) = g_1(-x)$ for $x \in R^2$ and $0 < \int_{R^2} g_1(x) dx < +\infty$, then, we have the following two results, which will be used later.

Lemma 1: (Theorem 3 in [30]): In the graph $G^{Poisson}(V', E(g_1))$,

$$
\lim_{\lambda \to +\infty} \frac{\sum_{k=1}^{+\infty} p_k(\lambda, r(\lambda))}{p_1(\lambda, r(\lambda))} = 1.
$$

This means that as $\lambda \to +\infty$, almost surely the origin lies in either an infinite-order cluster or an order-1 cluster (i.e., it is isolated).

Lemma 2: (Theorem 6.3 in [32]): When $\lambda \to +\infty$, there is at most one infinite cluster in $G^{Poisson}(V', E(g_1))$.

Based on Lemma 1 and 2, the following result can be easily obtained.

Lemma 3: As $\lambda \to +\infty$, the probability that the graph $G^{Poisson}(V', E(g_1))$ is connected is asymptotically the same as the probability that the graph $G^{Poisson}(V', E(g_1))$ has no isolated nodes, i.e.,

$$
\begin{aligned}
&\lim_{\lambda \to +\infty} \Pr[G^{Poisson}(V', E(g_1)) \text{ is connected}] \\
&= \lim_{\lambda \to +\infty} \Pr[G^{Poisson}(V', E(g_1)) \text{ has no isolated nodes}].
\end{aligned}
$$

In [30], it has been shown that

$$
p_1(\lambda, r(\lambda)) = \exp\left(-\lambda \int_{R^2} g_1(x) dx\right). \quad (8)
$$

Let $\lambda = n$. If $a_1 \pi r_0^2(n) = \frac{\log n + c(n)}{n}$, then we have $\int_{R^2} g_1(x) dx = a_1 \pi r_0^2(n) = \frac{\log n + c(n)}{n}$. So, from (8), the

probability of the origin to be isolated is:

$$p_1(\lambda, r(\lambda)) = \exp\left(-n \int_{\mathbb{R}^2} g_1(x)dx\right)$$
$$= \exp\left(-n \cdot \frac{\log n + c(n)}{n}\right)$$
$$= \exp\left(-\log n - c(n)\right)$$
$$= \frac{1}{n}e^{-c(n)}.$$

Let $E'(G)$ be the expected number of order-1 cluster and $p'(G)$ the probability that there is at least one order-1 cluster in graph G. Then,

$$p'\left(G^{Poisson}(V', E(g_1))\right) \leq E'\left(G^{Poisson}(V', E(g_1))\right)$$
$$= n \cdot p_1(\lambda, r(\lambda))$$
$$= e^{-c(n)}.$$

So, if $\lim_{n \to +\infty} c(n) = +\infty$, then $p'\left(G^{Poisson}(V', E(g_1))\right) \to 0$ as $n \to +\infty$.

Thus, from Lemma 3, we can obtain

$$\lim_{\lambda \to +\infty} \Pr\left[G^{Poisson}(V', E(g_1)) \text{ is connected}\right]$$
$$= 1 - p'\left(G^{Poisson}(V', E(g_1))\right)$$
$$= 1.$$

Since when the number of nodes n is sufficiently large, the difference between $G^{Poisson}(V', E(g_1))$ and $G(V', E(g_1))$ is negligible [14], then Theorem 2 directly follows.

Theorem 2: In DTDR networks, if $a_1 \pi r_0^2(n) = \frac{\log n + c(n)}{n}$ and $c(n) \to +\infty$, then the graph $G(V, E(g_1))$ is asymptotically connected with probability 1.

Combining Theorem 1 and Theorem 2, we arrive at our first main result.

Theorem 3: In DTDR networks, $G(V, E(g_1))$, with $a_1 \pi r_0^2(n) = \frac{\log n + c(n)}{n}$, is connected with probability 1 as $n \to +\infty$ if and only if $c(n) \to +\infty$.

B. DTOR Networks

In this subsection, we derive the necessary and sufficient condition for achieving asymptotic connectivity in DTOR networks. Once again, let r_m and r_s denote the transmission ranges, respectively, when the transmitter beamforms toward the receiver with the main lobe gain G_m and the side lobe gain G_s, respectively. By the model in (3), we can obtain

$$r_m = (G_m)^{\frac{1}{\alpha}} r_0, \tag{9}$$
$$r_s = (G_s)^{\frac{1}{\alpha}} r_0. \tag{10}$$

Fig. 5 shows the communication area of an arbitrary node i in DTOR networks. Let r denote the distance to node i, Area I and Area II the areas where $r_s < r \leq r_m$ and $0 < r \leq r_s$ in the main lobe direction, respectively, and Area III and Area IV the areas where $0 < r \leq r_s$ and $r_s < r \leq r_m$ in the side lobe directions, respectively. We observe that

(1) Each node can communicate with the nodes in Area I (S_1^{DO}) with probability of p_1^{DO}. Recall that in Section IV-A, the communication is bidirectionally symmetric,
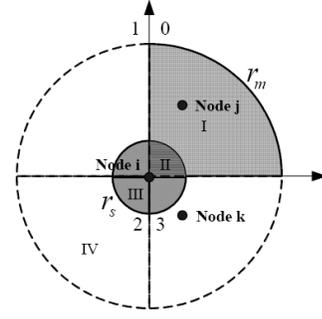
Fig. 5. The communication area of a node i in DTOR networks when directional antennas have four beam directions.

i.e., if node A can communicate with node B, then node B can also communicate with node A. However, in DTOR networks, the communication is bidirectionally asymmetric, i.e., if node A can communicate with node B, B may not necessarily be able to communicate with A. For example, if a node B is in Area I of node A, A is beamforming to B, but B is not beamforming toward A, then A can send packets to B but B cannot send packets to A. Specifically speaking, we define that if two nodes cannot be connected in any direction, the connectivity level is 0; and that if two nodes can be connected only in one direction, the connectivity level is 0.5; and that if two nodes can be connected in both directions, the connectivity level is 1. Thus we have

$$p_1^{DO} = \frac{1}{N} \cdot 1 + \frac{N-1}{N} \cdot \frac{1}{2} = \frac{N+1}{2N},$$
$$S_1^{DO} = \frac{1}{N}\pi(r_m^2 - r_s^2).$$

(2) Each node can communicate with the nodes in Area II (S_2^{DO}) with probability of p_2^{DO}, where

$$p_2^{DO} = 1,$$
$$S_2^{DO} = \frac{1}{N}\pi r_s^2.$$

(3) Each node can communicate with the nodes in Area III (S_3^{DO}) with probability of p_3^{DO}, where

$$p_3^{DO} = 1,$$
$$S_3^{DO} = \frac{N-1}{N}\pi r_s^2.$$

(4) Each node can communicate with the nodes in Area IV (S_4^{DO}) with probability of p_4^{DO}, where

$$p_4^{DO} = \frac{1}{N} \cdot \frac{1}{2} + \frac{N-1}{N} \cdot 0 = \frac{1}{2N},$$
$$S_4^{DO} = \frac{N-1}{N}\pi(r_m^2 - r_s^2).$$

Thus, the *effective area* S^{DO} of nodes in DTOR networks is

$$S^{DO} = \sum_{i=1}^{4} p_i^{DO} S_i^{DO}$$
$$= \frac{1}{N}\pi r_m^2 + \frac{N-1}{N}\pi r_s^2$$
$$= \left[\frac{1}{N}(G_m)^{\frac{2}{\alpha}} + \frac{N-1}{N}(G_s)^{\frac{2}{\alpha}}\right]\pi r_0^2.$$

Let $a_2 = [\frac{1}{N}(G_m)^{\frac{2}{\alpha}} + \frac{N-1}{N}(G_s)^{\frac{2}{\alpha}}]$. Then, $S^{DO} = a_2\pi r_0^2$. Following the procedures in Section IV-A, we can also use $g_2(x)$ to determine the edge set in the graph $G(V, E(g_2))$, where

$$g_2(x) = \begin{cases} 1 & \text{if } ||x||_2 \leq \sqrt{a_2}r_0 \\ 0 & \text{if } ||x||_2 > \sqrt{a_2}r_0 \end{cases}.$$

Thus, we can obtain the following results.

Theorem 4: In DTOR networks, if $a_2\pi r_0(n)^2 = \frac{\log n + c(n)}{n}$ and $\limsup_{n \to +\infty} c(n) < +\infty$, then $G(V, E(g_2))$ is asymptotically disconnected with positive probability.

Theorem 5: In DTOR networks, if $a_2\pi r_0^2(n) = \frac{\log n + c(n)}{n}$ and $c(n) \to +\infty$, then the graph $G(V, E(g_2))$ is asymptotically connected with probability 1.

In summary, we have for the DTOR networks

Theorem 6: In DTOR networks, $G(n, r_0(n))$, with $a_2\pi r_0^2(n) = \frac{\log n + c(n)}{n}$, is connected with probability 1 as $n \to +\infty$ if and only if $c(n) \to +\infty$.

C. OTDR Networks

In OTDR networks, the connection function $g_3(x)$ is the same as the connection function $g_2(x)$ in DTOR networks. So the *effective area* of a node in OTDR networks is the same as that in DTOR networks, i.e., $S^{OD} = S^{DO} = [\frac{1}{N}(G_m)^{\frac{2}{\alpha}} + \frac{N-1}{N}(G_s)^{\frac{2}{\alpha}}]\pi r_0^2$. Then, let $a_3 = [\frac{1}{N}(G_m)^{\frac{2}{\alpha}} + \frac{N-1}{N}(G_s)^{\frac{2}{\alpha}}]$, which is the same as a_2, and we only present the result here.

Theorem 7: In OTDR networks, $G(V, E(g_3))$, with $a_3\pi r_0^2(n) = \frac{\log n + c(n)}{n}$, is connected with probability 1 as $n \to +\infty$ if and only if $c(n) \to +\infty$.

V. MINIMIZING THE CRITICAL TRANSMISSION POWER

In Section IV, we have given the necessary and sufficient conditions for networks using directional antennas to achieve asymptotic connectivity, i.e.,

$$a_i\pi r_0^2(n) = \frac{\log n + c(n)}{n}, i = 1, 2, 3,$$
$$\text{as } n \to +\infty, c(n) \to +\infty, \quad (11)$$

where $a_1 = [\frac{1}{N}(G_m)^{\frac{2}{\alpha}} + \frac{N-1}{N}(G_s)^{\frac{2}{\alpha}}]^2$ in DTDR networks, and $a_2 = a_3 = [\frac{1}{N}(G_m)^{\frac{2}{\alpha}} + \frac{N-1}{N}(G_s)^{\frac{2}{\alpha}}]$ in DTOR and OTDR networks, as shown in Theorem 3, 6 and 7, respectively.

In addition, in [14], Gupta and Kumar show that when omnidirectional antennas are used, the necessary and sufficient conditions for OTOR networks to achieve asymptotic connectivity is

$$\pi r_0^2(n) = \frac{\log n + c(n)}{n}, \text{as } n \to +\infty, c(n) \to +\infty. \quad (12)$$

Comparing (11) with (12), we have

$$r_c^i = \frac{1}{\sqrt{a_i}}r_c, i = 1, 2, 3. \quad (13)$$

where r_c is the critical transmission range in OTOR networks, and $\{r_c^i, i = 1, 2, 3\}$ are the critical transmission ranges in DTDR, DTOR and OTDR networks, respectively. We observe that if $\{a_i, i = 1, 2, 3\}$ are greater than 1, the critical transmission ranges when directional antennas are used will be smaller than that when omnidirectional antennas are used.

Assume that the reception power needs to be greater than a constant threshold P_r^{thresh} in order for the receiver to correctly receive the message. According to the power propagation model introduced in Section III-B, we have

$$P_r^{thresh} = P_t\frac{C}{r_c^\alpha},$$

where C is a constant.

Let P_t, P_t^1, P_t^2, and P_t^3 denote the critical transmission powers in OTOR networks, DTDR networks, DTOR networks and OTDR networks, respectively. Then,

$$P_t^i = P_t\left(\frac{1}{a_i}\right)^{\alpha/2}, \quad (14)$$

where $i \in \{1, 2, 3\}$. So, in order to save power when using directional antennas, our objective is to minimize P^i, respectively, for $i = 1, 2, 3$, which is equivalent to maximizing $(a_i)^{\alpha/2}$, respectively, for $i = 1, 2, 3$, as shown below.

$$\text{Maximize } \left\{(a_i)^{\alpha/2}\right\}$$
$$\text{subject to}$$
$$0 \leq G_m \cdot a + G_s \cdot (1-a) \leq 1$$
$$G_m \geq 1, 0 \leq G_s \leq 1 \quad (15)$$

where $a_1 = [\frac{1}{N}(G_m)^{\frac{2}{\alpha}} + \frac{N-1}{N}(G_s)^{\frac{2}{\alpha}}]^2$, $a_2 = a_3 = [\frac{1}{N}(G_m)^{\frac{2}{\alpha}} + \frac{N-1}{N}(G_s)^{\frac{2}{\alpha}}]$, and $a = \frac{1}{2}(1 - \cos\frac{\pi}{N})$. Note that the first constraint is derived based on (2) by noticing $0 \leq \eta \leq 1$, and the second one is due to the characteristics of directional antennas as introduced in Section III-A. We have several cases:

(1) When $N = 2$, we have $a = \frac{1}{2}$ and $G_m + G_s \leq 2$. Then, for $\alpha > 2$, we obtain

$$(G_m)^{\frac{2}{\alpha}} + (G_s)^{\frac{2}{\alpha}} \leq (G_m + G_s)^{\frac{2}{\alpha}} \cdot 2^{1-\frac{2}{\alpha}} \leq 2,$$

according to Holder's inequality. Thus,

$$a_1 = \frac{1}{4}[(G_m)^{\frac{2}{\alpha}} + (G_s)^{\frac{2}{\alpha}}]^2 \leq 1,$$
$$a_2 = a_3 = \frac{1}{2}[(G_m)^{\frac{2}{\alpha}} + (G_s)^{\frac{2}{\alpha}}] \leq 1.$$

So, when $\alpha > 2$, all the maximum values can be achieved only when $G_m = G_s = 1$, i.e., when directional antennas work in the omni-directional mode. When $\alpha = 2$, it is obvious that the maximum values of a_1, a_2 and a_3 are all 1, and they can be achieved for any $G_s + G_m = 2$. As a result, when $N = 2$, the minimum critical transmission powers in DTDR, DTOR and OTDR networks are the same as the critical transmission power in OTOR networks.

(2) When $N > 2$, since $\sqrt{a_1} = a_2 = a_3 = \frac{1}{N}(G_m)^{\frac{2}{\alpha}} + \frac{N-1}{N}(G_s)^{\frac{2}{\alpha}}$, the three optimization problems presented in (15) can all be formulated as the same non-linear programming as

TABLE I
SOME CALCULATED VALUE OF $f(Gm, Gs, N, \alpha)$.

N	G_s	G_m	$f(Gm, Gs, N, \alpha)$
	0	6.83	0.8998
	0.001	6.82	0.9068
N = 4	0.01	6.77	0.9095
	0.1	6.25	1.0094
	0.5188	3.8046	1.0936 (Max)
	0	14.93	1.0105
	0.001	14.91	1.0182
N = 6	0.01	14.79	1.0428
	0.1	13.54	1.1261
	0.42	9.0784	1.1927 (Max)
	0	26.27	1.1047
	0.001	26.25	1.1128
N = 8	0.01	26.02	1.1383
	0.1	23.75	1.2212
	0.3632	17.0946	1.2749 (Max)

shown below:

$$\text{Maximize } \{f(G_m, G_s, N, \alpha)\}$$

subject to

$$A \cdot G \leq b$$

$$A = \begin{pmatrix} a & 1-a \\ -a & -(1-a) \\ -1 & 0 \\ 0 & -1 \\ 0 & 1 \end{pmatrix}, b = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix},$$

$$G = \begin{pmatrix} G_m \\ G_s \end{pmatrix}, \quad (16)$$

where $f(G_m, G_s, N, \alpha) = \frac{N-1}{N}(G_s)^{\frac{2}{\alpha}} + \frac{1}{N}(G_m)^{\frac{2}{\alpha}}$.

In wireless networks, the path loss exponent α is determined by the environment, so it can be considered as a known factor. Besides, the number of beams N of a directional antenna is a constant integer greater than 1. Thus, for each value of α and N, we can find optimal values of G_m and G_s to maximize $f(G_m, G_s, N, \alpha)$.

By setting α to 3, which is usually the case in urban areas [28], we calculate some "typical" values of $f(G_m, G_s, N, \alpha)$ when the number of beams N is 4, 6, and 8, respectively, as shown in Table I. Here, "typical" is referred to the settings where $G_s \leq$ -10 dBi, since usually we want to restrain the side lobe antenna gain in order to increase the main directional transmission range. However, this is not necessarily true if we want to achieve connectivity with a smaller transmission power.

From Table I, we can see that "typically" using directional antennas results in smaller critical transmission ranges, and hence smaller critical transmission powers than using omnidirectional antennas according to (13) and (14) since $f(G_m, G_s, N, \alpha) > 1$. From this table, we can also observe that the antenna pattern has great impacts on the value of $f(G_m, G_s, N, \alpha)$, and in turn the network connectivity.

So far, we have shown how to maximize $f(G_m, G_s, N, \alpha)$, with the maximum value denoted by $\max_{\{G_m, G_s\}}(f)$, given a certain value of N and α, respectively. Some values of $\max_{\{G_m, G_s\}}(f)$ when $N \in \{4, 6, 8\}$ and $\alpha = 3$ have already been shown in Table I. In the following, we show some more results of $\max_{\{G_m, G_s\}}(f)$.
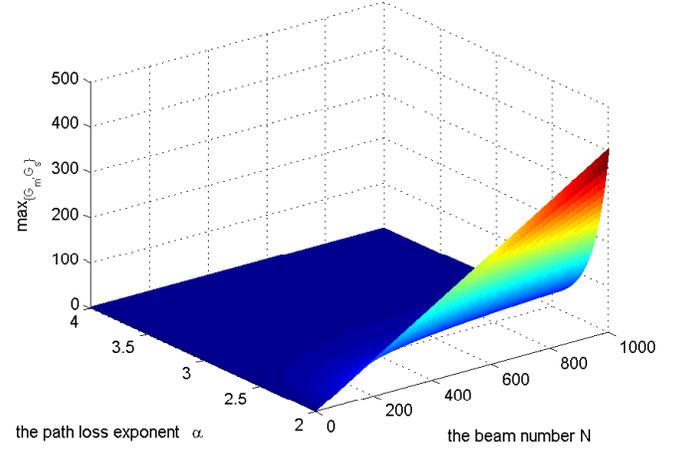
Fig. 6. The values of $\max_{\{G_m, G_s\}}(f)$ for $N \in [2, 1000]$, and $\alpha \in [2, 4]$.
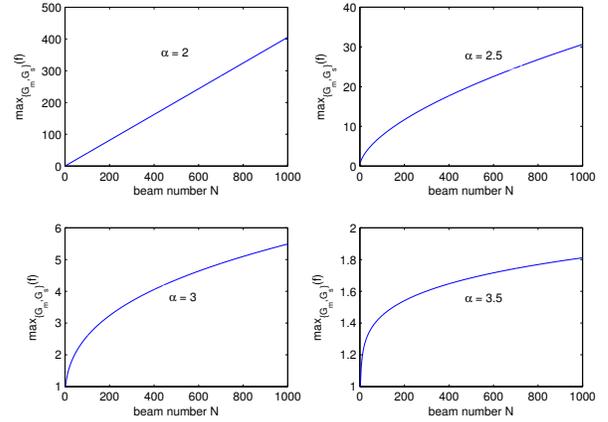
Fig. 7. The impact of path loss exponent on $\max_{\{G_m, G_s\}}(f)$.

The values of $\max_{\{G_m, G_s\}}(f)$ for $N \in [2, 1000]$ and $\alpha \in [2, 4]$ are shown in Fig. 6. Fig. 7 shows more clearly the impact of the path loss exponent on $\max_{\{G_m, G_s\}}(f)$. We observe that the value of $\max_{\{G_m, G_s\}}(f)$ can be approximated by a power function $y = pN^q$, where $p, q > 0$, and

$$q \begin{cases} = 1 & \text{when } \alpha = 2 \\ < 1 & \text{when } \alpha = 2.5, 3, 3.5 \end{cases}.$$

This shows that, with fixed α, $\max_{\{G_m, G_s\}}(f)$ increases as N increases, while with fixed N, $\max_{\{G_m, G_s\}}(f)$ decreases as α increases. The latter case is intuitively true because the increase of α implies the power propagation environment is getting worse and it will be more difficult to achieve network connectivity. Besides, we also find that when $N = 2$, $\max_{\{G_m, G_s\}}(f) = 1$ and when $N > 2$, $\max_{\{G_m, G_s\}}(f) > 1$, for all $\alpha \in [2, 4]$.

Furthermore, the value of the side lobe antenna gain G_s for $f(G_m, G_s, N, \alpha)$ to achieve the maximum is shown in Fig. 8. We observe that in order for $\max_{\{G_m, G_s\}}(f)$ to be large, the side lobe gain G_s has to approach 0 and the main lobe gain G_m has to approach $\frac{2}{1-\cos\frac{\pi}{N}}$.
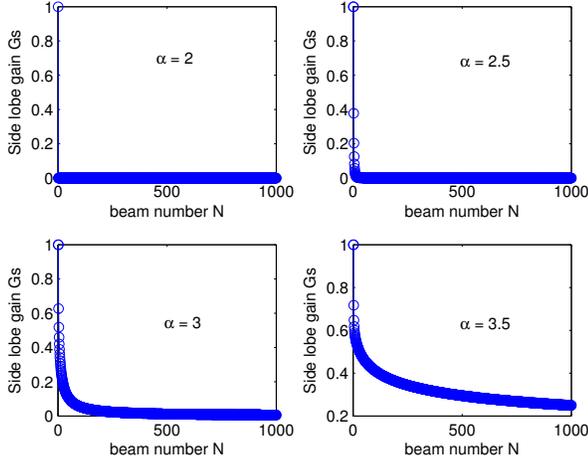
Fig. 8. The values of G_s to achieve $\max_{\{G_m,G_s\}}(f)$.

In addition, for any $N > 1$, we have

$$a_1 - a_2 = a_1 - a_3$$
$$= f^2(G_m, G_s, N, \alpha) - f(G_m, G_s, N, \alpha)$$
$$= f(G_m, G_s, N, \alpha) \cdot [f(G_m, G_s, N, \alpha) - 1].$$

which means that with the same parameter set (G_m, G_s, N, α), if the critical transmission range in DTOR or OTDR networks is smaller than that in OTOR networks, then the critical transmission range in DTDR networks is further smaller than that in DTOR and OTDR networks, and vice versa. In other words, with the same parameter set (Gm, Gs, N, α), if the critical transmission power in DTOR or OTDR networks is smaller than that in OTOR networks, then we can save even more power in DTDR networks, and vice versa.

As a result, we find that with the same number of beams N ($N > 2$) and the same path loss exponent α ($\alpha \in [2, 4]$), the minimum critical transmission power in DTDR networks is smaller than that in DTOR and OTDR networks, which is further smaller than that in OTOR networks. However, when the number of beams N is equal to 2, then with the same path loss exponent α, the minimum critical transmission power in DTDR, DTOR, and OTDR networks are all the same, which are equal to that in OTOR networks.

VI. SOME INSIGHTS ON OUR RESULTS

Recall that Gupta and Kumar [14] conclude that when omnidirectional antennas are used, $G(n, r_0(n))$ with $\pi r_0^2(n) = \frac{\log n + c(n)}{n}$ is connected with probability 1 as $n \to +\infty$ if and only if $c(n) \to +\infty$. This means that the expected number of neighbors of a node, which is $n\pi r_0^2(n)$, i.e., $\log n + c(n)$, has to approach infinity in order to achieve connectivity as $n \to +\infty$. We define $n\pi r_0^2(n)$ as *the critical expected number of neighbors* which is directly determined by critical transmission power.

In this paper, we have already shown that when directional antennas are used, $G(V, E(g_i))$ with $a_i\pi r_0^2(n) = \frac{\log n + c(n)}{n}$, where i is equal to 1, 2, or 3, corresponding to the cases of DTDR, DTOR or OTDR networks, respectively, are connected with probability 1 as $n \to +\infty$ if and only if $c(n) \to +\infty$.

This implies that in order to achieve connectivity, the expected number of neighbors of a node using directional antennas, which is $na_i\pi r_0^2(n)$, i.e., $\log n + c(n)$, still needs to approach infinity as $n \to +\infty$. However, the critical expected number of neighbors of a node, which is equal to $n\pi r_0^2(n)$ as defined before, i.e., $\frac{\log n + c(n)}{a_i}$, can be just $O(1)$ if a_i can be on the order of $\log n$. Therefore, when using directional antennas, we can save more transmission power if we can choose larger a_i. This is also shown clearly in (14).

In Section V, we show that $\max_{G_m,G_s}(f)$ increases as N increases in the range of $[2, 1000]$. Does $\max_{G_m,G_s}(f)$ keep increasing when $N > 1000$? We will try to answer this question by looking into the problem of maximizing $\max_{\{G_m,G_s\}}(f)$ with respect to the number of beams N to see how large $f(G_m, G_s, N, \alpha)$ can be, i.e.,

$$\max_N \left\{ \max_{\{G_m,G_s\}} f(G_m, G_s, N, \alpha) \right\}$$
$$\text{where } \{N : N \in \text{Integer}, N > 1 \},$$
$$f(G_m, G_s, N, \alpha) = \frac{N-1}{N}(G_s)^{\frac{2}{\alpha}} + \frac{1}{N}(G_m)^{\frac{2}{\alpha}}.$$

Since $f(G_m, G_s, N, \alpha)$ increases as either G_m increases or G_s increases. So the maximum of $f(G_m, G_s, N, \alpha)$ can be achieved only when $G_m \cdot a + G_s \cdot (1 - a) = 1$, where $a = \frac{1}{2}(1 - \cos\frac{\pi}{N})$, the same as that defined in (15). Thus, $f(G_m, G_s, N, \alpha)$ can be represented by $f(G_s)$, where

$$f(G_s) = \left\{ \frac{1}{N}[\frac{1}{a} - (\frac{1}{a} - 1)G_s]^{\frac{2}{\alpha}} + \frac{N-1}{N}(G_s)^{\frac{2}{\alpha}} \right\}. \quad (17)$$

(1) When $\alpha = 2$, we have

$$f(G_s) = \frac{1}{aN} + (1 - \frac{1}{aN})G_s.$$

As N goes large, we obtain

$$1 - \frac{1}{aN} = 1 - \frac{2}{N(1 - \cos\frac{\pi}{N})}$$
$$= 1 - \frac{1}{N\sin^2\frac{\pi}{2N}}$$
$$< 1 - \frac{4N}{\pi^2}$$
$$< 0, \quad (18)$$

since $\sin\frac{\pi}{2N} < \frac{\pi}{2N}$.

So, $\max_{\{G_m,G_s\}}(f) = 1/aN = \Omega(N)$, which can be achieved when $G_s = 0$, and $G_m = \frac{1}{a}$. Thus, we have

$$\max_N \left\{ \max_{\{G_m,G_s\}}(f) \right\} = \lim_{N \to +\infty} \left\{ \frac{1}{aN} \right\} \to +\infty.$$

(2) When $\alpha \in (2, 4)$, with N and α fixed, $\max_{\{G_m,G_s\}}(f)$ is achieved when $\partial f(G_s)/\partial G_s = 0$. Let G_s^* denote the result by solving the equation, then we obtain

$$G_s^* = \frac{b}{a + (1-a)b}, \quad (19)$$

where $a = \frac{1}{2}(1 - \cos\frac{\pi}{N})$, $b = [\frac{1-a}{a(N-1)}]^{\frac{\alpha}{2-\alpha}}$ for any $N > 1$.

Substituting (19) into (17), we obtain

$$\max_{\{G_m,G_s\}}(f) = f(G_s^*).$$

Then, we obtain that for $N \in (1, +\infty)$, $\alpha \in (2, 4)$,

$$\max_N \left\{ \max_{\{G_m, G_s\}} (f) \right\} = \lim_{N \to +\infty} \{f(G_s^*)\} \to +\infty,$$

which is achieved when $G_s = G_s^*$, where $\lim_{N \to +\infty} G_s^* = 0$.

(3) When $\alpha \in (0, 2)$ or $\alpha \in [4, +\infty)$, according to (17), we obtain

$$\begin{aligned}
f(G_m, G_s, N, \alpha)|_{G_s=0} &= \frac{1}{N} \left(\frac{1}{a} \right)^{\frac{2}{\alpha}} \\
&= \Omega \left(N^{\frac{4}{\alpha}-1} \right).
\end{aligned}$$

As a result, when $\alpha \in (0, 2)$, we have

$$\begin{aligned}
&\max_N \left\{ \max_{\{G_m, G_s\}} (f) \right\} \\
&\geq \max_N \{f(G_m, G_s, N, \alpha)|_{G_s=0}\} \to +\infty
\end{aligned}$$

as $N \to +\infty$. However, when $\alpha = 4$, we obtain

$$\max_N \left\{ \max_{\{G_m, G_s\}} (f) \right\} = 1.186,$$

which is achieved when $N \to +\infty$, and $G_s = 0.711$.

Moreover, when $\alpha > 4$,

$$\max_N \left\{ \max_{\{G_m, G_s\}} (f) \right\} \in (1, 1.186)$$

which is achieved when $N < +\infty$, and $0 < G_s < 1$.

Since $a_1 = f^2(G_m, G_s, N, \alpha)$, $a_2 = a_3 = f(G_m, G_s, N, \alpha)$, from the above, we obtain that for $i = 1, 2, 3$, respectively,

$$\max_N \left\{ \max_{\{G_m, G_s\}} (a_i) \right\} \begin{cases} \to +\infty & \text{if } \alpha \in (0, 4) \\ \in (1, 2) & \text{if } \alpha \in [4, +\infty) \end{cases}.$$

In conclusion, we show that for any $\alpha \in (0, 4)$, $\{a_i, i = 1, 2, 3\}$ can be made very large by adjusting the directional antenna pattern, i.e., (G_m, G_s), when N is large. This means that if with some transmission power, the critical expected number of neighbors of a node is only $O(1)$ by using omnidirectional antennas, we can still make the network connected by using directional antennas with the same transmission power. Thus, we can save a lot of power with directional antennas because in OTOR networks, the transmission power needs to be set such that each node has $O(\log n)$ expected neighboring nodes. Besides, for $\alpha \in [4, +\infty)$, the maximum values of $\{a_i, i = 1, 2, 3\}$ are in the range $(1, 2)$. Thus, we can still save power by using directional antennas.

VII. AN EXTENDED CASE

In Section IV, we derive the necessary and sufficient conditions for achieving asymptotic connectivity with the assumption that each node in the network randomly beamforms in one of the N beam directions. However, if there exists a schedule according to which two neighboring nodes can be guaranteed to beamform to each other to carry out the communication, the connectivity problem would be rather different. In this section, we discuss this new connectivity problem in details.

We still classify WANETs using directional antennas into four categories, i.e., DTDR networks, DTOR networks, OTDR networks, and OTOR networks. Then, the effective areas of DTDR, DTOR, and OTDR networks, denoted by S_{DD}, S_{DO}, and S_{OD}, respectively, are equal to:

$$\begin{aligned}
S_{DD} &= \pi r_{mm}^2 = (G_m)^{\frac{4}{\alpha}} \pi r_0^2, \\
S_{DO} &= S_{OD} = \pi r_m^2 = (G_m)^{\frac{2}{\alpha}} \pi r_0^2,
\end{aligned}$$

where r_{mm} and r_m are defined in (4) and (9), respectively.

Following the process in Section IV, we obtain the following result.

Theorem 8: If there exists a schedule according to which two neighboring nodes can be guaranteed to beamform to each other to carry out the transmission, with $b_i \pi r_0^2(n) = \frac{\log n + c(n)}{n}$, $i = 1, 2, 3$, for DTDR, DTOR, and OTDR networks, respectively, are connected with probability 1 as $n \to +\infty$ if and only if $c(n) \to +\infty$, where $b_1 = (G_m)^{\frac{4}{\alpha}}$, $b_2 = b_3 = (G_m)^{\frac{2}{\alpha}}$.

Note that b_1, b_2, and b_3 are larger than a_1, a_2, and a_3, respectively, which means the critical transmission ranges are smaller than those in previous case, and so are the critical transmission powers.

As shown in (15), we have $0 \leq G_m \cdot a + G_s \cdot (1 - a) \leq 1$ where $a = \frac{1}{2}(1 - \cos \frac{\pi}{N})$. Thus,

$$\max(b_1) = \left(\frac{1}{a} \right)^{\frac{4}{\alpha}} = \Omega \left(N^{\frac{8}{\alpha}} \right),$$

$$\max(b_2) = \max(b_3) = \left(\frac{1}{a} \right)^{\frac{2}{\alpha}} = \Omega \left(N^{\frac{4}{\alpha}} \right).$$

As a result, $\max(b_i) \to +\infty$ as $N \to +\infty$ for $i = 1, 2, 3$, respectively, which is achieved when $G_s = 0$, and $G_m = \frac{1}{a}$. This means, for any α, if with a transmission power level so that the critical expected number of neighbors of a node is only $O(1)$ when using omnidirectional antennas, we can still make the network connected by using directional antennas with the same transmission power.

Besides, we also notice that at current stage of technology the beam number N is very small. Can our claim that the critical number of neighbors can be just $O(1)$ to achieve connectivity still hold? Denote $f(G_m, G_s, N, \alpha)$ in (16), which is contained in Theorem 3, 6, and 7, by $f1$, and $(G_m)^{\frac{2}{\alpha}}$ in Theorem 8 by $f2$. In order for our conclusion to hold, we can make $f1$ and $f2$ be $\log n$ in DTOR and OTDR networks, and $\sqrt{\log n}$ in DTDR networks, where n is the number of nodes in the network. We show the values of $f1$, $f2$, and $\log n$ in Fig. 9 when $\alpha = 2$, and $n = 100, 1000, 10000$. We can clearly observe that a beam number smaller than 25 is sufficient for our conclusion to hold even in a large wireless network with 10000 nodes. Furthermore, we realize that as the number of nodes n goes larger, we need to increase N accordingly because N is a monotonically increasing function of n. Thus, for large n, our claim can only hold if the beam number N is also made large, which we hope to be true as the directional antenna technology progresses.

VIII. CONCLUSION

In this paper, we study the connectivity problem in WANETs using directional antennas. Since transmission and
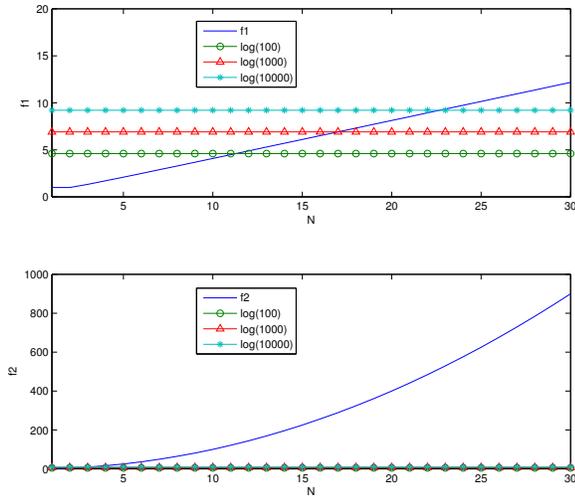
Fig. 9. The values of $f1$ and $f2$ compared to $\log n$ when $n = 100, 1000, 10000$ and $\alpha = 2$.

reception schemes have significant impacts on the network connectivity, we classify the networks into four categories: DTDR networks, DTOR networks, OTDR networks, and OTOR networks.

Under the assumption that each node in the network randomly beamforms in one of the N beam directions, in outdoor environments with the path loss exponent $\alpha \in [2, 4)$, we obtain the following conclusion:

(C1) When the beam number N is 2, the minimum critical transmission powers in DTDR, DTOR, and OTDR networks are all the same, which are equal to the critical transmission power in OTOR networks.

(C2) When the number of beams N is larger than 2, the minimum critical transmission power in DTDR networks is smaller than that in DTOR and OTDR networks, which is further smaller than that in OTOR networks.

(C3) In DTDR, DTOR and OTDR networks, we can still make the network connected by using directional antennas even when the critical expected number of neighbors is just $O(1)$.

Moreover, we show that (C3) actually holds for any $\alpha \in (0, 4)$.

Besides, we also study the connectivity problem when there exists a schedule according to which two neighboring nodes can be guaranteed to beamform to each other to communicate. In this case, we show that (C3) actually holds for any $\alpha \in (0, +\infty)$.

Finally, we realize that in real deployment, the number of nodes is usually limited, and hence the reliable estimation on critical transmission range in the case with limited number of nodes is also interesting and challenging. We will investigate this problem in our future research.

REFERENCES

[1] I. F. Akyildiz, W. Su, Y. Sankarasubramaniam, and E. Cayirci, "Wireless sensor networks: A survey," *Computer Networks: The International Journal of Computer and Telecommunications Networking*, vol. 38, no. 4, pp. 393–422, March 2002.

[2] I. F. Akyildiz, X. Wang, and W. Wang, "Wireless mesh networks: A survey," *Computer Networks*, vol. 47, no. 4, pp. 445–487, March 2005.

[3] R. Choudhury and N. Vaidya, "Deafness: A mac problem in ad hoc networks when using directional antennas," in *Proceedings of the 12th IEEE International Conference on Network Protocols (ICNP)*, Berlin, Germany, October 2004, pp. 283–292.

[4] R. R. Choudhury, X. Yang, N. Vaidya, and R. Ramanathan, "Using directional antennas for medium access control in ad hoc networks," in *Proceeding of ACM MobiCom*, Atlanta, Georgia, USA, September 2002.

[5] H. Gossain, C. M. Cordeiro, D. Cavalcanti, and D. P. Agrawal, "The deafness problems and solutions in wireless ad hoc networks using directional antennas," in *IEEE Workshop on Wireless Ad Hoc and Sensor Networks*, Dallas, Texas, USA, November 2004.

[6] Y. Ko, V. Shankarkumar, and N. Vaidya, "Medium access control protocols using directional antennas in ad hoc networks," in *Proceeding of the IEEE International Conference on Computer Communications (INFOCOM'00)*, Tel Aviv, Israel, March 2000.

[7] V. Kolar, S. Tilak, and N. Abu-Ghazaleh, "Avoiding head of line blocking in directional antenna," in *Proceedings of the 29th Annual IEEE International Conference on Local Computer Networks (ICN)*, Tampa, Florida, USA, November 2004.

[8] T. Korakis, G. Jakllari, and L. Tassiulas, "A mac protocol for full exploitation of directional antennas in ad-hoc wireless networks," in *Proceeding of ACM MobiHoc*, Annapolis, Maryland, USA, June 2003.

[9] P. Li, H. Zhai, and Y. Fang, "A directional mac protocol for ad hoc networks," in *Proceedings of IEEE Military Communications Conference (Milcom'06)*, Washington, DC, USA, October 2006.

[10] ——, "Sdmac: Selectively directional mac protocol for wireless mobile ad hoc networks," *ACM Wireless Networks*, October 31, 2007.

[11] A. Nasipuri, S. Ye, and R. Hiromoto, "A mac protocol for mobile ad hoc networks using directional antennas," in *Proceedings of the IEEE Wireless Communications and Networking Conference (WCNC)*, vol. 3, Chicago, Illinois, USA, September 2000.

[12] M. Taikai, J. Martin, A. Ren, and R. Bagrodia, "Directional virtual carrier sensing for directional antennas in mobile ad hoc networks," in *Proceeding of ACM MobiHoc*, Lausanne, Switzerland, June 2002.

[13] K.-K. Yap, W.-L. Yeow, M. Motani, and C.-K. Tham, "Simple directional antennas: Improving performance in wireless multihop networks," in *Proceeding of the IEEE International Conference on Computer Communications (INFOCOM'06)*, Barcelona, Spain, Apr. 2006.

[14] P. Gupta and P. Kumar, "Critical power for asymptotic connectivity in wireless networks," *Stochastic Analysis, Control, Optimization and Applications*, vol. A Volume in Honor of W.H. Fleming, pp. 547–566, 1998.

[15] P. Li, C. Zhang, and Y. Fang, "Asymptotic connectivity in wireless networks using directional antennas," in *Proceedings of ICDCS*, Toronto, Canada, June 2007.

[16] K. Madsen, F. Fitzek, G. Schulte, and R. Prasad, "Connectivity probability of wireless ad hoc networks: Definition, evaluation, comparison," *Special Issue of the International Journal on Wireless Personal Communications*, vol. 35, no. 1-2, pp. 135–151, October 2005.

[17] O. Dousse, F. Baccelli, and P. Thiran, "Impact of interferences on connectivity in ad hoc networks," in *Proceeding of the IEEE International Conference on Computer Communications (INFOCOM'03)*, San Francisco, California, USA, April 2003.

[18] P. Wan and C. Yi, "Asymptotic critical transmission ranges for connectivity in wireless ad hoc networks with bernoulli nodes," in *Proceeding of IEEE WCNC*, New Orleans, Louisiana, USA, March 2005.

[19] C. Yi, P. Wan, X. Li, and O. Frieder, "Asymptotic distribution of the number of isolated nodes in wireless ad hoc networks with bernoulli nodes," *IEEE Transactions on Communications*, vol. 54, no. 3, pp. 510-517, March 2006.

[20] P. Balister, B. Bollobas, A. Sarkar, and S. Kumar, "Reliable density estimates for coverage and connectivity in thin strips of finite length," in *Proceedings of ACM MobiCom*, Montreal, Quebec, Canada, September 2007.

[21] E. Kranakis, D. Krizanc, and E. Williams, "Directional versus omnidirectional antennas for energy consumption and k-connectivity of networks of sensors," in *Proceedings of OPODIS*, Grenoble, France, December 2004.

[22] J. Diaz, V. Sanwalani, M. Serna, and P. Spirakis, "Chromatic number of random scaled sector graphs," *Theoretical Computer Science*, vol. 349, no. 1, pp. 40–51, 2005.

[23] C. Bettstetter, C. Hartmann, and C. Moser, "How does randomized beamforming improve the connectivity of ad hoc networks?" in *Proceedings of the IEEE International Conference on Communications (ICC)*, vol. 5, Seoul, Korea, May 2005.

[24] S. Yi, Y. Pei, and S. Kalyanaraman, "On the capacity improvement of ad hoc wireless networks using directional antennas," in *Proceeding of ACM MobiHoc*, Annapolis, Maryland, USA, June 2003.

[25] J. Liberti and T. Rappaport, *Smart Antennas for Wireless Communications*. Prentice-Hall PTR, 1999.

[26] R. Janaswamy, *Radiowave Propagation and Smart Antennas for Wireless Communications*. Boston, USA: Kluwer Academic Publishers, 2001.

[27] R. Ramanathan, "On the performance of ad hoc networks with beam-forming antennas," in *Proceedings of ACM MobiHoc*, Long Beach, California, USA, October 2001.

[28] T. Rappaport, *Wireless Communications: Principles and Practice (Second Edition)*. Prentice-Hall PTR, 2002.

[29] E. Sevick, P. Monson, and J. Ottino, "Clustering and percolation in assemblies of anisotropic particles: Perturbation theory and monte carlo simulation," *Physical Review A*, vol. 38, no. 10, pp. 5376–5383, November 1988.

[30] M. D. Penrose, "On a continuum percolation model," *Advances in Applied Probability*, vol. 23, no. 3, pp. 536–556, 1991.

[31] ——, "The longest edge of the random minimal spanning tree," *The Annals of Applied Probability*, vol. 7, no. 2, pp. 340–361, 1997.

[32] R. Meester and R. Roy, *Continuum Percolation*. Cambridge, UK: Cambridge University Press, 1996.

Yuguang "Michael" Fang (IEEE, S'92-M'97-SM'99-F'08) received a Ph.D. degree in Systems Engineering from Case Western Reserve University in January 1994 and a Ph.D degree in Electrical Engineering from Boston University in May 1997. He was an assistant professor in the Department of Electrical and Computer Engineering at New Jersey Institute of Technology from July 1998 to May 2000. He then joined the Department of Electrical and Computer Engineering at University of Florida in May 2000 as an assistant professor, got an early promotion to an associate professor with tenure in August 2003 and to a full professor in August 2005. He holds a University of Florida Research Foundation (UFRF) Professorship from 2006 to 2009. He has published over 200 papers in refereed professional journals and conferences. He received the National Science Foundation Faculty Early Career Award in 2001 and the Office of Naval Research Young Investigator Award in 2002. He is the recipient of the Best Paper Award in IEEE International Conference on Network Protocols (ICNP) in 2006 and the recipient of the IEEE TCGN Best Paper Award in the IEEE High-Speed Networks Symposium, IEEE Globecom in 2002. Dr. Fang is also active in professional activities. He is an Fellow of IEEE and a member of ACM. He has served on several editorial boards of technical journals including IEEE Transactions on Communications, IEEE Transactions on Wireless Communications, IEEE Transactions on Mobile Computing and ACM Wireless Networks. He has been actively participating in professional conference organizations such as serving as the Steering Committee Co-Chair for QShine, the Technical Program Vice-Chair for IEEE INFOCOM'2005, Technical Program Symposium Co-Chair for IEEE Globecom'2004, and a member of Technical Program Committee for IEEE INFOCOM (1998, 2000, 2003-2009).

Pan Li (IEEE, S'06 / ACM'07) received his BS in Electrical Engineering from Huazhong University of Science and Technology, Wuhan, China, in 2005. He is working towards his Ph.D. degree in the Department of Electrical and Computer Engineering at University of Florida. His research interests include capacity and connectivity analysis, medium access control, routing algorithms, and cross-layer design in wireless networks. He is a student member of the IEEE and the ACM.

Chi Zhang (IEEE, S'06) received the B.E. and M.E. degrees in Electrical Engineering from Huazhong University of Science and Technology, Wuhan, China, in July 1999 and January 2002, respectively. Since September 2004, he has been working towards the Ph.D. degree in the Department of Electrical and Computer Engineering at University of Florida, Gainesville, Florida, USA. His research interests are network and distributed system security, wireless networking, and mobile computing, with emphasis on mobile ad hoc networks, wireless sensor networks, wireless mesh networks, and heterogeneous wired/wireless networks.